Examples-ForHomalg

Examples for the GAP Package homalg

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[BLH20]

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Installation of the ExamplesForHomalg Package

To install this package just extract the package's archive file to the GAP `pkg` directory.

By default the ExamplesForHomalg package is not automatically loaded by GAP when it is installed. You must load the package with

```
LoadPackage("ExamplesForHomalg");
```

before its functions become available.

Please, send us an e-mail if you have any questions, remarks, suggestions, etc. concerning this package. Also, I would be pleased to hear about applications of this package.

Mohamed Barakat and Simon Görtzen.

Chapter 3

Examples

3.1 Spectral Filtrations

3.1.1 ExtExt

This is Example B.2 in [Bar09].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
gap> wmat := HomalgMatrix( "[ \
> x*y,  y*z,    z,        0,         0,    \
> x^3*z,x^2*z^2,0,        x*z^2,     -z^2, \
> x^4,  x^3*z,  0,        x^2*z,     -x*z, \
> 0,    0,      x*y,      -y^2,      x^2-1,\
> 0,    0,      x^2*z,    -x*y*z,    y*z,  \
> 0,    0,      x^2*y-x^2,-x*y^2+x*y,y^2-y \
> ]", 6, 5, Qxyz );
<A 6 x 5 matrix over an external ring>
gap> W := LeftPresentation( wmat );
<A left module presented by 6 relations for 5 generators>
gap> Y := Hom( Qxyz, W );
<A right module on 5 generators satisfying yet unknown relations>
gap> SetInfoLevel( InfoWarning, 0 );
gap> F := InsertObjectInMultiFunctor( Functor_Hom_for_fp_modules, 2, Y, "TensorY" );
<The functor TensorY for f.p. modules and their maps over computable rings>
gap> SetInfoLevel( InfoWarning, 1 );
gap> G := LeftDualizingFunctor( Qxyz );;
gap> II_E := GrothendieckSpectralSequence( F, G, W );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 3 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
Level 0:
```

```
 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s s s s
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 * * * *
 . * * *
 . . . *
---------
Level 2:

 * * s s
 * * * *
 . * * *
 . . . *
---------
Level 3:

 * s s s
 * s s s
 . . s *
 . . . *
---------
```

```
Level 4:

 s s s s
 . s s s
 . . s s
 . . . s
gap> filt := FiltrationBySpectralSequence( II_E, 0 );
<An ascending filtration with degrees [ -3 .. 0 ] and graded parts:

0:   <A non-zero left module presented by yet unknown relations for 23 generator\
s>
  -1:   <A non-zero left module presented by 37 relations for 22 generators>
  -2:   <A non-zero left module presented by 31 relations for 10 generators>
  -3:   <A non-zero left module presented by 33 relations for 5 generators>
of
<A non-zero left module presented by 102 relations for 37 generators>>
gap> ByASmallerPresentation( filt );
<An ascending filtration with degrees [ -3 .. 0 ] and graded parts:
   0:   <A non-zero left module presented by 26 relations for 16 generators>
  -1:   <A non-zero left module presented by 30 relations for 14 generators>
  -2:   <A non-zero left module presented by 18 relations for 7 generators>
  -3:   <A non-zero left module presented by 12 relations for 4 generators>
of
<A non-zero left module presented by 48 relations for 20 generators>>
gap> m := IsomorphismOfFiltration( filt );
<A non-zero isomorphism of left modules>
```

3.1.2 Purity

This is Example B.3 in [Bar09].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
gap> wmat := HomalgMatrix( "[ \
> x*y,  y*z,    z,        0,         0,    \
> x^3*z,x^2*z^2,0,        x*z^2,     -z^2, \
> x^4,  x^3*z,  0,        x^2*z,     -x*z, \
> 0,    0,      x*y,      -y^2,      x^2-1,\
> 0,    0,      x^2*z,    -x*y*z,    y*z,  \
> 0,    0,      x^2*y-x^2,-x*y^2+x*y,y^2-y \
> ]", 6, 5, Qxyz );
<A 6 x 5 matrix over an external ring>
gap> W := LeftPresentation( wmat );
<A left module presented by 6 relations for 5 generators>
gap> filt := PurityFiltration( W );
<The ascending purity filtration with degrees [ -3 .. 0 ] and graded parts:

0:   <A codegree-[ 1, 1 ]-pure rank 2 left module presented by 3 relations for 4\
 generators>

-1:   <A codegree-1-pure grade 1 left module presented by 4 relations for 3 gene\
rators>
```

```
-2:   <A cyclic reflexively pure grade 2 left module presented by 2 relations fo\
r a cyclic generator>

-3:   <A cyclic reflexively pure grade 3 left module presented by 3 relations fo\
r a cyclic generator>
of
<A non-pure rank 2 left module presented by 6 relations for 5 generators>>
gap> W;
<A non-pure rank 2 left module presented by 6 relations for 5 generators>
gap> II_E := SpectralSequence( filt );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 3 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s . . .
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
```

```
---------
Level 1:

 * * * *
 * * * *
 . * * *
 . . . *
---------
Level 2:

 s . . .
 * s . .
 . * * .
 . . . *
---------
Level 3:

 s . . .
 * s . .
 . . s .
 . . . *
---------
Level 4:

 s . . .
 . s . .
 . . s .
 . . . s

gap> m := IsomorphismOfFiltration( filt );
<A non-zero isomorphism of left modules>
gap> IsIdenticalObj( Range( m ), W );
true
gap> Source( m );
<A left module presented by 12 relations for 9 generators (locked)>
gap> Display( last );
0,  0,x, -y,0,1, 0,    0,  0,
x*y,0,-z,0, 0,0, 0,    0,  0,
x^2,0,0, -z,1,0, 0,    0,  0,
0,  0,0, 0, y,-z,0,    0,  0,
0,  0,0, 0, 0,x, -y,   -1, 0,
0,  0,0, 0, x,0, -z,   0,  -1,
0,  0,0, 0, 0,-y,x^2-1,0,  0,
0,  0,0, 0, 0,0, 0,    z,  0,
0,  0,0, 0, 0,0, 0,    y-1,0,
0,  0,0, 0, 0,0, 0,    0,  z,
0,  0,0, 0, 0,0, 0,    0,  y,
0,  0,0, 0, 0,0, 0,    0,  x

Cokernel of the map

Q[x,y,z]^(1x12) --> Q[x,y,z]^(1x9),
```

```
currently represented by the above matrix
gap> Display( filt );
Degree 0:

0,  0,x, -y,
x*y,0,-z,0,
x^2,0,0, -z

Cokernel of the map

Q[x,y,z]^(1x3) --> Q[x,y,z]^(1x4),

currently represented by the above matrix
----------
Degree -1:

y,-z,0,
0,x, -y,
x,0, -z,
0,-y,x^2-1

Cokernel of the map

Q[x,y,z]^(1x4) --> Q[x,y,z]^(1x3),

currently represented by the above matrix
----------
Degree -2:

Q[x,y,z]/< z, y-1 >
----------
Degree -3:

Q[x,y,z]/< z, y, x >
gap> Display( m );
1,    0,      0,  0,   0,
0,    1,      0,  0,   0,
0,    -y,     -1, 0,   0,
0,    -x,     0,  -1,  0,
-x^2,-x*z,   0,  -z,  0,
0,    0,      x,  -y,  0,
0,    0,      0,  0,   -1,
0,    0,      x^2,-x*y,y,
-x^3,-x^2*z,0,  -x*z,z

the map is currently represented by the above 9 x 5 matrix
```

3.1.3 A3_Purity

This is Example B.4 in [Bar09].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
gap> A3 := RingOfDerivations( Qxyz, "Dx,Dy,Dz" );
Q[x,y,z]<Dx,Dy,Dz>
gap> nmat := HomalgMatrix( "[ \
> 3*Dy*Dz-Dz^2+Dx+3*Dy-Dz,          3*Dy*Dz-Dz^2,      \
> Dx*Dz+Dz^2+Dz,                    Dx*Dz+Dz^2,        \
> Dx*Dy,                            0,                 \
> Dz^2-Dx+Dz,                       3*Dx*Dy+Dz^2,      \
> Dx^2,                             0,                 \
> -Dz^2+Dx-Dz,                      3*Dx^2-Dz^2,       \
> Dz^3-Dx*Dz+Dz^2,                  Dz^3,              \
> 2*x*Dz^2-2*x*Dx+2*x*Dz+3*Dx+3*Dz+3,2*x*Dz^2+3*Dx+3*Dz\
> ]", 8, 2, A3 );
<A 8 x 2 matrix over an external ring>
gap> N := LeftPresentation( nmat );
<A left module presented by 8 relations for 2 generators>
gap> filt := PurityFiltration( N );
<The ascending purity filtration with degrees [ -3 .. 0 ] and graded parts:
   0:   <A zero left module>

-1:   <A cyclic reflexively pure grade 1 left module presented by 1 relation for\
 a cyclic generator>

-2:   <A cyclic reflexively pure grade 2 left module presented by 2 relations fo\
r a cyclic generator>

-3:   <A cyclic reflexively pure grade 3 left module presented by 3 relations fo\
r a cyclic generator>
of
<A non-pure grade 1 left module presented by 8 relations for 2 generators>>
gap> II_E := SpectralSequence( filt );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 2 ] each consisting of left modules at bidegrees [ -4 .. 0 ]x
[ 0 .. 3 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -4 .. 0 ] ]
---------
Level 0:

 * * * *
 . * * *
 . . * *
 . . . *
 . . . *
---------
Level 1:

 * * * *
```

```
 . . . .
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s . . .
 . . . .
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -4 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * * *
 . . * * *
 . . . * *
 . . . . *
---------
Level 1:

 * * * * *
 . . * * *
 . . . * *
 . . . . .
---------
Level 2:

 . s . . .
 . . s . .
 . . . s .
 . . . . .
gap> m := IsomorphismOfFiltration( filt );
<A non-zero isomorphism of left modules>
gap> IsIdenticalObj( Range( m ), N );
true
gap> Source( m );
<A left module presented by 6 relations for 3 generators (locked)>
gap> Display( last );
Dx,1/3,1/216*x,
0, Dy, -1/144,
0, Dx, 1/48,
0, 0,  Dz,
0, 0,  Dy,
0, 0,  Dx
```

```
Cokernel of the map

R^(1x6) --> R^(1x3), ( for R := Q[x,y,z]<Dx,Dy,Dz> )

currently represented by the above matrix
gap> Display( filt );
Degree 0:

0
----------
Degree -1:

Q[x,y,z]<Dx,Dy,Dz>/< Dx >
----------
Degree -2:

Q[x,y,z]<Dx,Dy,Dz>/< Dy, Dx >
----------
Degree -3:

Q[x,y,z]<Dx,Dy,Dz>/< Dz, Dy, Dx >
gap> Display( m );
1,                      1,
3*Dz+3,                 3*Dz,
144*Dz^2-144*Dx+144*Dz,144*Dz^2

the map is currently represented by the above 3 x 2 matrix
```

3.1.4 TorExt-Grothendieck

This is Example B.5 in [Bar09].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
gap> wmat := HomalgMatrix( "[ \
> x*y,  y*z,    z,        0,         0,    \
> x^3*z,x^2*z^2,0,        x*z^2,     -z^2, \
> x^4,  x^3*z,  0,        x^2*z,     -x*z, \
> 0,    0,      x*y,      -y^2,      x^2-1,\
> 0,    0,      x^2*z,    -x*y*z,    y*z,  \
> 0,    0,      x^2*y-x^2,-x*y^2+x*y,y^2-y \
> ]", 6, 5, Qxyz );
<A 6 x 5 matrix over an external ring>
gap> W := LeftPresentation( wmat );
<A left module presented by 6 relations for 5 generators>
gap> F := InsertObjectInMultiFunctor( Functor_TensorProduct_for_fp_modules, 2, W, "TensorW" );
<The functor TensorW for f.p. modules and their maps over computable rings>
gap> G := LeftDualizingFunctor( Qxyz );;
gap> II_E := GrothendieckSpectralSequence( F, G, W );
<A stable cohomological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 3 ]>
```

```
gap> Display( II_E );
The associated transposed spectral sequence:

a cohomological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s s s s
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a cohomological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 * * * *
 . * * *
 . . . *
---------
Level 2:

 * * s s
 * * * *
 . * * *
 . . . *
```

```
---------
Level 3:

 * s s s
 . s s s
 . . s *
 . . . s
---------
Level 4:

 s s s s
 . s s s
 . . s s
 . . . s
gap> filt := FiltrationBySpectralSequence( II_E, 0 );
<A descending filtration with degrees [ -3 .. 0 ] and graded parts:

-3:   <A non-zero cyclic torsion left module presented by yet unknown relations \
for a cyclic generator>
  -2:   <A non-zero left module presented by 17 relations for 6 generators>
  -1:   <A non-zero left module presented by 27 relations for 12 generators>
   0:   <A non-zero left module presented by 13 relations for 10 generators>
of
<A left module presented by yet unknown relations for 49 generators>>
gap> ByASmallerPresentation( filt );
<A descending filtration with degrees [ -3 .. 0 ] and graded parts:

-3:   <A non-zero cyclic torsion left module presented by 3 relations for a cycl\
ic generator>
  -2:   <A non-zero left module presented by 12 relations for 4 generators>
  -1:   <A non-zero left module presented by 21 relations for 8 generators>
   0:   <A non-zero left module presented by 11 relations for 10 generators>
of
<A non-zero left module presented by 27 relations for 14 generators>>
gap> m := IsomorphismOfFiltration( filt );
<A non-zero isomorphism of left modules>
```

3.1.5 TorExt

This is Example B.6 in [Bar09].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
gap> wmat := HomalgMatrix( "[ \
> x*y,  y*z,    z,        0,         0,    \
> x^3*z,x^2*z^2,0,        x*z^2,     -z^2, \
> x^4,  x^3*z,  0,        x^2*z,     -x*z, \
> 0,    0,      x*y,      -y^2,      x^2-1,\
> 0,    0,      x^2*z,    -x*y*z,    y*z,  \
> 0,    0,      x^2*y-x^2,-x*y^2+x*y,y^2-y \
> ]", 6, 5, Qxyz );
<A 6 x 5 matrix over an external ring>
```

```
gap> W := LeftPresentation( wmat );
<A left module presented by 6 relations for 5 generators>
gap> P := Resolution( W );
<A right acyclic complex containing 3 morphisms of left modules at degrees
[ 0 .. 3 ]>
gap> GP := Hom( P );
<A cocomplex containing 3 morphisms of right modules at degrees [ 0 .. 3 ]>
gap> FGP := GP * P;
<A cocomplex containing 3 morphisms of left complexes at degrees [ 0 .. 3 ]>
gap> BC := HomalgBicomplex( FGP );
<A bicocomplex containing left modules at bidegrees [ 0 .. 3 ]x[ -3 .. 0 ]>
gap> p_degrees := ObjectDegreesOfBicomplex( BC )[1];
[ 0 .. 3 ]
gap> II_E := SecondSpectralSequenceWithFiltration( BC, p_degrees );
<A stable cohomological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 3 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a cohomological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * * *
 * * * *
 * * * *
 * * * *
---------
Level 1:

 * * * *
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s s s s
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a cohomological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * *
```

```
 * * * *
 * * * *
 * * * *
---------
Level 1:

 * * * *
 * * * *
 * * * *
 * * * *
---------
Level 2:

 * * s s
 * * * *
 . * * *
 . . . *
---------
Level 3:

 * s s s
 . s s s
 . . s *
 . . . s
---------
Level 4:

 s s s s
 . s s s
 . . s s
 . . . s
gap> filt := FiltrationBySpectralSequence( II_E, 0 );
<A descending filtration with degrees [ -3 .. 0 ] and graded parts:

-3:   <A non-zero cyclic torsion left module presented by yet unknown relations \
for a cyclic generator>
  -2:   <A non-zero left module presented by 15 relations for 6 generators>
  -1:   <A non-zero left module presented by 27 relations for 13 generators>
   0:   <A non-zero left module presented by 13 relations for 10 generators>
of
<A left module presented by yet unknown relations for 31 generators>>
gap> ByASmallerPresentation( filt );
<A descending filtration with degrees [ -3 .. 0 ] and graded parts:

-3:   <A non-zero cyclic torsion left module presented by 3 relations for a cycl\
ic generator>
  -2:   <A non-zero left module presented by 11 relations for 4 generators>
  -1:   <A non-zero left module presented by 23 relations for 9 generators>
   0:   <A non-zero left module presented by 11 relations for 10 generators>
of
<A non-zero left module presented by 24 relations for 12 generators>>
gap> m := IsomorphismOfFiltration( filt );
```

```
<A non-zero isomorphism of left modules>
```

3.1.6 CodegreeOfPurity

This is Example B.7 in [Bar09].

Example
```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
gap> vmat := HomalgMatrix( "[ \
> 0,  0,  x,-z, \
> x*z,z^2,y,0,  \
> x^2,x*z,0,y   \
> ]", 3, 4, Qxyz );
<A 3 x 4 matrix over an external ring>
gap> V := LeftPresentation( vmat );
<A non-torsion left module presented by 3 relations for 4 generators>
gap> wmat := HomalgMatrix( "[ \
> 0,  0,  x,-y, \
> x*y,y*z,z,0,  \
> x^2,x*z,0,z   \
> ]", 3, 4, Qxyz );
<A 3 x 4 matrix over an external ring>
gap> W := LeftPresentation( wmat );
<A non-torsion left module presented by 3 relations for 4 generators>
gap> Rank( V );
2
gap> Rank( W );
2
gap> ProjectiveDimension( V );
2
gap> ProjectiveDimension( W );
2
gap> DegreeOfTorsionFreeness( V );
1
gap> DegreeOfTorsionFreeness( W );
1
gap> CodegreeOfPurity( V );
[ 2 ]
gap> CodegreeOfPurity( W );
[ 1, 1 ]
gap> filtV := PurityFiltration( V );
<The ascending purity filtration with degrees [ -2 .. 0 ] and graded parts:

0:   <A codegree-[ 2 ]-pure rank 2 left module presented by 3 relations for 4 ge\
nerators>
  -1:   <A zero left module>
  -2:   <A zero left module>
of
<A codegree-[ 2 ]-pure rank 2 left module presented by 3 relations for 4 gener\
ators>>
gap> filtW := PurityFiltration( W );
<The ascending purity filtration with degrees [ -2 .. 0 ] and graded parts:
```

```
0:   <A codegree-[ 1, 1 ]-pure rank 2 left module presented by 3 relations for 4\
 generators>
  -1:   <A zero left module>
  -2:   <A zero left module>
of
<A codegree-[ 1, 1 ]-pure rank 2 left module presented by 3 relations for 4 ge\
nerators>>
gap> II_EV := SpectralSequence( filtV );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 2 ]>
gap> Display( II_EV );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 2 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * *
 * * *
 * * *
 . * *
---------
Level 1:

 * * *
 . . .
 . . .
 . . .
---------
Level 2:

 s . .
 . . .
 . . .
 . . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 2 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
---------
Level 1:
```

```
 * * * *
 * * * *
 . . * *
---------
Level 2:

 * . . .
 * . . .
 . . * *
---------
Level 3:

 * . . .
 . . . .
 . . . *
---------
Level 4:

 . . . .
 . . . .
 . . . s
gap> II_EW := SpectralSequence( filtW );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 2 ]>
gap> Display( II_EW );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 2 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * *
 * * *
 . * *
 . . *
---------
Level 1:

 * * *
 . . .
 . . .
 . . .
---------
Level 2:

 s . .
 . . .
 . . .
 . . .
```

```
Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 2 ] ]
---------
Level 0:

 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 . * * *
 . . . *
---------
Level 2:

 * . . .
 . * . .
 . . . *
---------
Level 3:

 * . . .
 . . . .
 . . . *
---------
Level 4:

 . . . .
 . . . .
 . . . s
```

3.1.7 HomHom

This corresponds to the example of Section 2 in [BR06].

Example

```
gap> R := HomalgRingOfIntegersInExternalGAP( ) / 2^8;
Z/( 256 )
gap> Display( R );
<A residue class ring>
gap> M := LeftPresentation( [ 2^5 ], R );
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> Display( M );
Z/( 256 )/< |[ 32 ]| >
gap> M;
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> _M := LeftPresentation( [ 2^3 ], R );
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> Display( _M );
```

```
Z/( 256 )/< |[ 8 ]| >
gap> _M;
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> alpha2 := HomalgMap( [ 1 ], M, _M );
<A "homomorphism" of left modules>
gap> IsMorphism( alpha2 );
true
gap> alpha2;
<A homomorphism of left modules>
gap> Display( alpha2 );
[ [  1 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
gap> M_ := Kernel( alpha2 );
<A cyclic left module presented by yet unknown relations for a cyclic generato\
r>
gap> alpha1 := KernelEmb( alpha2 );
<A monomorphism of left modules>
gap> seq := HomalgComplex( alpha2 );
<An acyclic complex containing a single morphism of left modules at degrees
[ 0 .. 1 ]>
gap> Add( seq, alpha1 );
gap> seq;
<A sequence containing 2 morphisms of left modules at degrees [ 0 .. 2 ]>
gap> IsShortExactSequence( seq );
true
gap> seq;
<A short exact sequence containing 2 morphisms of left modules at degrees
[ 0 .. 2 ]>
gap> Display( seq );
-------------------------
at homology degree: 2
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  8 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 1
Z/( 256 )/< |[ 32 ]| >
-------------------------
[ [  1 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 0
```

```
Z/( 256 )/< |[ 8 ]| >
-------------------------
gap> K := LeftPresentation( [ 2^7 ], R );
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> L := RightPresentation( [ 2^4 ], R );
<A cyclic right module on a cyclic generator satisfying 1 relation>
gap> triangle := LHomHom( 4, seq, K, L, "t" );
<An exact triangle containing 3 morphisms of left complexes at degrees
[ 1, 2, 3, 1 ]>
gap> lehs := LongSequence( triangle );
<A sequence containing 14 morphisms of left modules at degrees [ 0 .. 14 ]>
gap> ByASmallerPresentation( lehs );
<A non-zero sequence containing 14 morphisms of left modules at degrees
[ 0 .. 14 ]>
gap> IsExactSequence( lehs );
false
gap> lehs;
<A non-zero left acyclic complex containing
14 morphisms of left modules at degrees [ 0 .. 14 ]>
gap> Assert( 0, IsLeftAcyclic( lehs ) );
gap> Display( lehs );
-------------------------
at homology degree: 14
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  4 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 13
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 12
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 11
Z/( 256 )/< |[ 4 ]| >
-------------------------
```

```
[ [  4 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 10
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 9
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 8
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  4 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 7
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 6
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 5
```

```
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  4 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 4
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 3
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 2
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  8 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 1
Z/( 256 )/< |[ 16 ]| >
-------------------------
[ [  1 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 0
Z/( 256 )/< |[ 8 ]| >
-------------------------
```

3.2 Commutative Algebra

3.2.1 Eliminate

Example

```
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z,l,m";
Q[x,y,z,l,m]
gap> var := Indeterminates( R );
[ x, y, z, l, m ]
gap> x := var[1];; y := var[2];; z := var[3];; l := var[4];; m := var[5];;
gap> L := [ x*m+l-4, y*m+l-2, z*m-l+1, x^2+y^2+z^2-1, x+y-z ];
[ x*m+l-4, y*m+l-2, z*m-l+1, x^2+y^2+z^2-1, x+y-z ]
gap> e := Eliminate( L, [ l, m ] );
<A non-zero right regular 3 x 1 matrix over an external ring>
gap> Display( e );
4*y+z,
4*x-5*z,
21*z^2-8
gap> I := LeftSubmodule( e );
<A torsion-free (left) ideal given by 3 generators>
gap> Display( I );
4*y+z,
4*x-5*z,
21*z^2-8

A (left) ideal generated by the 3 entries of the above matrix
gap> J := LeftSubmodule( "x+y-z, -2*z-3*y+x, x^2+y^2+z^2-1", R );
<A torsion-free (left) ideal given by 3 generators>
gap> I = J;
true
```

References

[Bar09] Mohamed Barakat. Spectral filtrations via generalized morphisms. (arXiv:0904.0240), 2009. 6, 8, 11, 14, 16, 19

[BLH20] Mohamed Barakat and Markus Lange-Hegermann. *The* homalg *package – A homological algebra* GAP4 *meta-package for computable Abelian categories*, 2007–2020. (https://homalg-project.github.io/pkg/homalg). 4

[BR06] Mohamed Barakat and Daniel Robertz. homalg: First steps to an abstract package for homological algebra. In *Proceedings of the X meeting on computational algebra and its applications - EACA 2006*, pages 29–32, Sevilla, Spain, September 2006. 22

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